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Evolution of Fields in a Second Order Phase Transition

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Abstract

We analyse the evolution of scalar and gauge fields during a second order phase transition using a Langevin equation approach. We show that topological defects formed during the phase transition are stable to thermal fluctuations. Our method allows the field evolution to be followed throughout the phase transition, for both expanding and non-expanding Universes. The results verify the Kibble mechanism for defect formation during phase transitions.

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1 Introduction

Why there is so little anti-matter in the Universe, and why the matter coalesced in the way it did are two of the major problems facing cosmology. Predictably, both have attracted a great deal of attention spawning a panoply of explanations and theories. Some of these theories involve objects known as topological defects[1], regions of trapped primordial vacuum, an example of which is the cosmic string. A string approach is an appealing one since it can be used to address both questions: the wake left by strings moving through the Universe can produce fluctuations which may lead to the accretion of matter into large scale structures[2][3], whilst their interaction with particles and the decay of string loops can provide mechanisms leading to baryon number violation[4] and the observed matter bias[5]. Hence, it is important to understand how these strings form, both to predict how many we can expect to have been created and how likely the above processes are. The formation of topological defects is thought to proceed via the Kibble mechanism[1].

Modern particle physics and the hot big bang model suggest that as the Universe cooled it underwent several phase transitions in which the symmetry of the vacuum was broken into a successively smaller, and smaller group. During such a transition, it is possible for fields to acquire non-zero vacuum expectation values. How they do this depends on the order of the transition.

If we consider a transition where a U(1) symmetry is broken, then following the transition all points in space will have a physically identical, non-zero vacuum expectation value, the only variation being in the difference in phase between any two points.

By causality, we expect the phases to be uncorrelated on distances greater than the horizon length, and so there is a finite probability that the phase along a closed path through a number of Horizon volumes will wind through some multiple of 2π - an indication that the loop contains a string. In practice two points do not need to be separated by a horizon length for their phases to be uncorrelated. This should also be true if they are a thermal correlation length (defined later) apart; usually a considerably smaller distance than the horizon size.

This is the Kibble mechanism, and it relies upon the so-called geodesic rule, which is that in passing between two domains of different phase, the phase will follow the shortest route. In the global case this has been verified both numerically[6] and experimentally[7][8], though, for a local symmetry it has been argued[10] that the presence of gauge fields may influence the path the phase takes, and may actually prevent it from following a 'geodesic'. More recently, however, work has been done suggesting that, despite this, the geodesic rule holds in the local case for a first order phase transition[9].

As the temperature falls below the critical temperature, for a while it is still possible for thermal fluctuations to restore the broken symmetry, and hence erase any topologically interesting configuration present at the time. The point at which it is thought that this ceases to be possible is referred to as the Ginzburg temperature, T_G, and is found by equating the free energy with the thermal energy (for such a restoring fluctuation will have a high probability while the former is considerably less than the latter). Brandenberger and Davis[11] demonstrated that given certain constraints on the parameters, the ratio of fluctuations in the scalar field to the background is less than one beneath a temperature just below the Ginzburg temperature, regardless of whether gauge fields are, or are not, present. This means that topologically non-trivial configurations arising from thermal fluctuations will become stable to such fluctuations just under the Ginzburg temperature. This adds weight to the arguments

in favour of the Kibble mechanism.

However, since Brandenberger and Davis considered a linearized model, only valid for the short time immediately following a spinodal decomposition, before non-linear effects start to dominate, their analysis only holds for early times. To study the evolution of the fields at later times it is necessary to include the non-linearities. A flexible way to do this is to study the Langevin equation associated with the classical field equations[12]. This is the purpose of this paper.

By studying the Langevin equation for the system, we derive an equation for the probability distribution of the fields, $P(\phi_i, A_i^\mu, t)$ which we use to analyse the evolution of the expectation values of the classical fields coupled to a thermal bath. This enables us to study not only the stability of configurations to fluctuations at and below the Ginzburg temperature, but also the long time evolution of the fields. The flexibility of this method is demonstrated by the ease with which it is modified to include the expansion of the Universe.

Our method seems to be the only one that allows the study of the effect that thermal fluctuations have on the development of the field, and the stability of defects formed, *throughout* the phase transition. Other methods either concentrate on the start of the phase transition, or near its completion.

2 Global Symmetry

Although our ultimate aim is to study the case of an expanding universe with broken local symmetry, it is beneficial, for several reasons, to start with the more straightforward case of a global, non-expanding model. Firstly, since the Kibble mechanism has already been verified for this case, we know that any topologically non-trivial configurations present should be stable to fluctuations below T_G, and hence we have a benchmark to check our results against. Although this provides a useful test of our method, it is by no means a proof of its validity. Secondly, since the amount of algebra involved is very dependent on the number of fields present, the global, non-expanding case provides the simplest, and hence clearest, demonstration of the method used throughout.

Consider the $U(1)$ toy model

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - V(|\phi|^2) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1}$$

where ϕ is a complex scalar field, A_μ is the U(1) gauge connection (taken to be zero for now) and $D_\mu = \partial_\mu - ieA_\mu$. We adopt an effective potential of the form

$$V(|\phi|^2) = \frac{\lambda}{4}(|\phi|^2 - \eta^2)^2 + \frac{\tilde{\lambda}}{2} T^2 |\phi|^2, \tag{2}$$

where $\tilde{\lambda} = (4\lambda + 6e^2)/12$ and the temperature dependence reflects the fluctuations on a scale smaller than some correlation length, defined later. For sufficiently high temperatures this is symmetric about a global minimum at zero. However, as the temperature passes through some critical temperature, $T_C = (\lambda/\tilde{\lambda})^{\frac{1}{2}} \eta$, the system undergoes a second order phase transition, breaking the U(1) symmetry, with new minima appearing at $|\phi|^2 = \eta^2 (1 - T^2/T_C^2)$. Any two points in the new vacuum will now have non-zero vacuum expectation values of equal moduli but random phase.

Setting $\phi = \rho \exp(i\alpha)$, our equations of motion for ρ and α are

$$\begin{aligned} \partial_\mu \partial^\mu \rho - (\partial_\mu \alpha \partial^\mu \alpha)\rho + \tfrac{dV(\rho^2)}{d\rho^2}\rho = 0, \\ \partial_\mu \partial^\mu \alpha + 2\partial^\mu \alpha \partial^\mu \rho / \rho = 0. \end{aligned} \tag{3}$$

We assume that α is time independent and varies spatially over some length-scale $2\pi/k_\alpha$. Setting $R = \dot{\rho}$ we obtain

$$\dot{R} + F\rho = 0 \quad , \quad \dot{\rho} - R = 0, \tag{4}$$

where

$$F = \left(k_C^2 - \frac{\lambda}{2}\eta^2 + \frac{\tilde{\lambda}}{2}T^2 - (k_\alpha \alpha)^2 + \frac{\lambda}{2}\langle \rho^2 \rangle \right), \tag{5}$$

and k_C is explained below. Note that we have replaced $\lambda\rho^3/2$ with $\lambda\langle\rho^2\rangle\rho/2$ c.f the mean square approximation to make the resulting equations more accessible.

For the purpose of this analysis we consider an initial configuration varying spatially on a scale of the correlation length. Since we do not want to become embroiled in a discussion of effects due to fluctuations on scales shorter than the thermal correlation length, $\xi_G = 1/[\eta\sqrt{\lambda(1 - T_G^2/T_C^2)}]$, we consider a coarse-grained field where we have integrated out all modes associated with such. This leads to the effective potential mentioned earlier. Hence, if we were to perform a Fourier decomposition, then it would be of the form $\rho = \sum_{k \leq k_c} \rho_k \exp(i\underline{k}.\underline{x})$ for some $k_c \sim 1/\xi_G$. We also assume that the mode corresponding to k_c dominates, (which we later show to be self-consistent) and investigate a configuration with a length scale $2\pi/k_C$. By (3) we see that the earlier assumption that $\dot{\alpha} = 0$ requires $k_\alpha = 2k_c$.

To incorporate thermal fluctuations into our model, we modify (3) such that the equations describing the evolution of R and ρ over some small time interval δt are

$$R(t + \delta t) = R(t) - \delta t F \rho(t) + \delta R, \tag{6}$$

$$\rho(t + \delta t) = \rho(t) + \delta t R(t) + \delta\rho, \tag{7}$$

where $\delta\rho$ and δR are the thermal fluctuations in ρ and R respectively.

Defining $\rho_{\delta t} = \rho(t + \delta t)$, $\rho = \rho(t)$, and similarly $R_{\delta t}$ and R, we can write

$$\begin{aligned} P(\rho_{\delta t}, R_{\delta t}, t + \delta t) = & \int d(\delta\rho) d(\delta R) P_1(\delta\rho) P_2(\delta R) \\ & \times P(\rho_{\delta t} - \delta t R_{\delta t} - \delta\rho, R_{\delta t} + \delta t F \rho_{\delta t} - \delta R, t) \\ & \times \frac{\partial}{\partial \rho_{\delta t}}(\rho_{\delta t} - \delta t R_{\delta t} - \delta\rho) \\ & \times \frac{\partial}{\partial R_{\delta t}}(R_{\delta t} + \delta t F \rho_{\delta t} - \delta R), \end{aligned} \tag{8}$$

where P_1 and P_2 are the probability measures for $\delta\rho$ and δR respectively.

Since $\delta\rho$ and δR are random fluctuations, we may assume that $\langle\delta\rho\rangle$=$\langle\delta R\rangle = 0$. Expanding the integrands as Taylor series, we find, after considerable algebra, that

$$\begin{aligned} P_{\delta t} &= P - \delta t \frac{\partial}{\partial\rho}(RP) + \delta t \frac{\partial}{\partial R}(F\rho P) \\ &\quad + \frac{\partial}{\partial\rho}\left(\frac{1}{2}\langle\delta\rho^2\rangle\frac{\partial P}{\partial\rho}\right) + \frac{\partial}{\partial R}\left(\frac{1}{2}\langle\delta R^2\rangle\frac{\partial P}{\partial R}\right) \end{aligned} \tag{9}$$

from which, assuming that $\delta\rho$ and δR are independent of ρ and R, we obtain the following differential equation for P:

$$\begin{aligned} \frac{\partial P}{\partial t} &= -\frac{\partial}{\partial\rho}(RP) + \frac{\partial}{\partial R}(F\rho P) \\ &\quad + \frac{1}{2}\frac{\langle\delta\rho^2\rangle}{\delta t}\frac{\partial P}{\partial\rho} + \frac{1}{2}\frac{\langle\delta R^2\rangle}{\delta t}\frac{\partial P}{\partial R}. \end{aligned} \tag{10}$$

One interpretation of this equation is as follows. If we move the first two terms on the right hand side over to the left, then we have a full derivative of P. Liouville's theorem states that for a closed system this derivative should be zero. However, our system is coupled to a thermal bath and so there is a flow of probability between the two, as demonstrated by the two non-zero noise terms, due to the bath, on the right hand side.

This equation is clearly a rather forbidding equation to solve analytically. However, we can use it to derive equations governing the quadratic moments

$$\langle\rho^2\rangle = \int d\rho dR P(\rho, R, t)\rho^2, \quad \langle\rho R\rangle = \int d\rho dR P(\rho, R, t)\rho R, \quad \langle R^2\rangle = \int d\rho dR P(\rho, R, t) R^2.$$

We choose to investigate these moments because their equations form a closed system with the mean field approximation we have taken.

As a modification we set

$$u = \langle\rho^2\rangle/\eta^2 \quad , \quad v = \langle\rho R\rangle/\eta^3 \quad , \quad w = \langle R^2\rangle/\eta^4 \quad , \quad \tau = \eta t, \tag{11}$$

since this normalises u, v and w, and gives the equations for the moments in a form where the relative sizes of terms are much more apparent;

$$\dot{u} = 2v + \delta_1, \tag{12}$$

$$\dot{v} = -f_o u - \frac{1}{2}\lambda u^2 + w^2, \tag{13}$$

$$\dot{w} = -2f_o v - \lambda uv + \delta_2, \tag{14}$$

where $\cdot$ denotes $\frac{d}{d\tau}$,

$$\delta_1 = \frac{1}{\eta^3}\frac{\langle\rho^2\rangle}{\delta t}, \qquad \delta_2 = \frac{1}{\eta^5}\frac{\langle R^2\rangle}{\delta t} \tag{15}$$

are the fluctuations and

$$f_o = \left(\frac{k^2}{\eta^2} - \frac{\lambda}{2} + \frac{\tilde{\lambda}}{2}\frac{T^2}{\eta^2} - \frac{(k_\alpha \alpha)^2}{\eta^2}\right). \tag{16}$$

It should be noted that these equations are even nastier than they look at first glance, since f_o contains a term proportional to T^2. However, by assuming that the temperature varies at a much slower rate than the fields (something which we will see is self-consistent later on) it is possible to make some progress analytically.

For the time being, we consider the case when fluctuations are absent, and treat f_o as constant over a small time period. After a bit of substitution, we integrate the equations to obtain

$$\dot{u}^2 = -\lambda u^3 - 4 f_o u^2 + \Lambda_1 u + \Lambda_2, \tag{17}$$

where $\Lambda_1 = 4(w_i + f_o u_i) + \lambda u_i^2$, $\Lambda_2 = 4(v_i^2 - u_i w_i)$ and u_i, v_i and w_i are the initial values of u, v and w.

Let the three roots of the polynomial on the right hand side be $\mu_1 > \mu_2 > \mu_3$. Hence

$$\mu_1\mu_2\mu_3 = \tfrac{4\Lambda_2}{\lambda}, \quad \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1 = -\tfrac{4\Lambda_1}{\lambda}, \quad \mu_1 + \mu_2 + \mu_3 = -\tfrac{4 f_o}{\lambda}. \tag{18}$$

Taking initial conditions such that, at T_G, $u_i = 1 - t_C/t_G = \lambda^2/[g(1+\lambda^2/g)]$, which is the minimum of the effective potential at that time, and $v_i = w_i = 0$, we find $\Lambda_2 = 0$, implying that one of the roots is zero. Furthermore, Λ_1 and f_o are found to be negative, so the two non-zero roots must be positive. Since $\dot{u}^2$ is seen to be positive between μ_1 and μ_2 we expect u to oscillate between these two.

We can also make an estimate of the time period of these oscillations, t_P, since

$$t_P = \frac{4}{\sqrt{\lambda\mu_1}} K(\kappa), \tag{19}$$

where

$$K(\kappa) = \frac{\sqrt{\mu_1}}{2}\int_{\mu_2}^{\mu_1} \frac{du}{\sqrt{(\mu_1 - u)(u - \mu_2)(u - \mu_3)}},$$

is the first complete elliptic integral and $\kappa = \sqrt{[(\mu_1 - \mu_2)/\mu_1]}$. These results closely agree with the corresponding numerical calculations. However, they are only valid for very small fluctuations, and so we turn to a numerical approach for a more detailed analysis.

In order to study the effect of fluctuations on the evolution of the fields, it is necessary to make some estimate of the fluctuation terms. To do this we imagine ϕ coupled to some other field, ψ, in thermal equilibrium, via an extra term in the Lagrangian of the form

$$\mathcal{L}_I = \frac{1}{2} g |\phi|^2 |\psi|^2. \tag{20}$$

By comparing the resulting equations of motion with those already obtained we find that

$$\delta\rho = 0 \quad , \quad \delta R = g\rho\psi^2\delta t.$$

Since ψ is in thermal equilibrium, we also have that $\psi \sim T$, with corresponding number density $n_\psi = 1.202 g_* T^3/\pi^2$ where g_* is the number of internal degrees of freedom (107 for a Grand Unified Theory). Taking δt to be a typical interaction time, such that $\delta t \sim n_\psi^{-1}\sigma_I^{-1}$, where $\sigma_I =\sim gk^{-2}$ is the interaction cross-section, we find the following approximation for the fluctuation terms

$$\delta_1 = 0 \quad , \quad \delta_2 = g\left(\frac{k^2}{\eta^2}\right)\frac{T}{\eta}u. \tag{21}$$

As expected, the size of the fluctuations decreases with temperature.

Before we can carry out a calculation, we need to address the problem of choice of parameter values. Since the model we are considering is a global one, we see from the definition of the potential that we must have $\tilde{\lambda} = \lambda/3$. We now demonstrate why we are justified in assuming that the mode corresponding to k_C dominates. Figure 1 shows the evolution of the quadratic moment u, (corresponding to $\langle\rho^2\rangle$) from the Ginzburg time onwards, where we have taken $\lambda = 0.1$, fluctuation coupling, $g = \lambda/3$ and a range of different wavelengths. The most obvious feature is that the mode varying with wavenumber k_C dominates those with longer wavelengths, consistent with our earlier assumption. However, one may be slightly alarmed at the fact that at least two of the curves look like they have no intention of converging to one (corresponding to $\langle\rho^2\rangle \to \eta^2$) as one might expect. The reasons for this are twofold, and both somewhat of our own creation. The first is that in assuming that ρ and α vary spatially with fixed wavenumber, we alter the value of ρ for which $\dot{\rho} = 0$ since we have essentially added two terms onto the derivative of the potential. This has the effect of raising the equilibrium value of ρ. The second is that we have to make an arbitrary choice of α (taken throughout as $\alpha = 1$), which effectively scales k_α, and so has a similar effect to the first. Conversely it could also be used to tune the expected equilibrium value to one by choosing a sufficiently small value of α.

This may raise questions over the validity of this method for studying the evolution of fields. However, the evolution of the fields is not qualitatively changed by taking different values of k_C, k_α or α and so we argue that, as an approximation, our approach is still of interest.

The potential we are using is, unfortunately, only a one loop approximation, and hence is not valid above the Ginzburg temperature where higher loops dominate. Our simulation therefore must run from the Ginzburg time onwards, and so we can only investigate the stability of string configurations to thermal fluctuations and not their formation. We also only consider the case of a GUT phase transition, since we expect one at a lower temperature, such as the Electro-weak phase transition for instance, to be qualitatively the same, but slower due to the larger value of t_C. We take the coupling λ to be between 0.01 and 1, and $g \leq \lambda$.

Figure 2 shows the effect of the two couplings, g and λ. The former controls the size of fluctuations, and it is seen, in 2a and 2b, that the larger g, the quicker the rise of the lower bound, so fluctuations are, bizarrely, actually seen to help stabilise configurations, on

average, by damping oscillations in the field. They also cause the upper bound to rise at a faster rate though this is not as pronounced, nor as important to the stability of domain structures.

Fig.2c reveals that decreasing λ decreases the frequency of oscillations, and also the asymptotic value for u. The latter is because $k_C \propto 1/\xi_G \propto \sqrt{\lambda}$ and we have already noted that the value of k_C effects the limiting value. Finally, Fig.2d demonstrates that the effect of fluctuations decreases dramatically with λ.

We see then that, since all curves move away from zero, any topologically non-trivial configuration is stable from the Ginzburg temperature onwards, though the fields may take a long time to reach their equilibrium values. We also note that, in all cases considered, the oscillations occur on a much smaller timescale than the evolution towards the equilibrium value; consistent with our earlier assumption.

3 The Effect of Gauge Fields

That the configurations formed in the above transition are stable against thermal fluctuations is nothing new. The Kibble mechanism for the global case is already well accepted, since one can argue in favour of the geodesic rule just by demanding that the path followed minimizes the energy density. However, the presence of gauge fields may undermine this, since their presence in the gradient energy, $(D_\mu\phi)(D^\mu\phi)^\dagger$, may make it equally favourable, energetically, to follow a longer path.

Luckily, the method used for studying the global case works equally well in the local one, the only drawback being a significant increase in the amount of algebra that has to be done. We start by writing the equations of motion in the form

$$
\begin{aligned}
\partial_\mu\partial^\mu\rho - e^2(q_\mu q^\mu)\rho + \frac{\partial V(\rho^2)}{\partial\rho^2}\rho &= 0, \\
\partial_\mu\partial^\mu q^\nu + 2e^2\rho^2 q^\nu + \frac{1}{e}\partial_\mu\partial^\mu\partial^\nu\alpha &= 0,
\end{aligned}
$$

where $q^\nu = A^\nu - \frac{1}{e}\partial^\nu\alpha$ (Note that this is gauge invariant). Setting $Q^\mu = \dot{q}^\mu$, $R = \dot{\rho}$ and $\Delta^\nu = \frac{1}{e}\partial_\mu\partial^\mu\partial^\nu\alpha$ for convenience, we find

$$
\begin{array}{rclcrcrcc}
R(t+\delta t) &=& R(t) &-& \delta t F_2(t)\rho(t) &+& \delta R, && \\
\rho(t+\delta t) &=& \rho(t) &+& \delta t R(t) &+& \delta\rho, && \\
Q^\mu(t+\delta t) &=& Q^\mu(t) &-& \delta t G(t) q^\mu(t) &+& \delta Q^\mu &-& \delta t\Delta^\mu, \\
q^\mu(t+\delta t) &=& q^\mu(t) &+& \delta t Q^\mu(t) &+& \delta q^\mu, &&
\end{array}
\tag{22}
$$

where

$$
\begin{aligned}
F_2 &= \left(k^2 - \frac{\lambda}{2}\eta^2 + \frac{\tilde{\lambda}}{2}T^2 + \frac{\lambda}{2}\langle\rho^2\rangle - e^2\langle q^2\rangle\right), \\
G &= \left(k^2 + 2e^2\langle\rho^2\rangle\right),
\end{aligned}
$$

and δR, $\delta\rho$, δQ^μ, δq^μ are the thermal fluctuations.

Proceeding in exactly the same manner as before, we arrive, after some very unpleasant algebra, at the equation for $P(\rho, R, q^\mu, Q^\mu, t)$,

$$\begin{aligned}\frac{\partial P}{\partial t} &= -\frac{\partial}{\partial \rho}(RP) + \frac{\partial}{\partial R}(F_2 \rho P) - \frac{\partial}{\partial q_\mu}(Q_\mu P) + \frac{\partial}{\partial Q_\mu}(G q_\mu P) + \Delta_\mu \frac{\partial P}{\partial Q_\mu} \\ &\quad + \frac{1}{2}\frac{\langle \delta\rho^2 \rangle}{\delta t}\frac{\partial^2 P}{\partial \rho^2} + \frac{1}{2}\frac{\langle \delta R^2 \rangle}{\delta t}\frac{\partial^2 P}{\partial R^2} \\ &\quad + \frac{1}{2}\frac{\langle \delta q_\mu \delta q^\mu \rangle}{\delta t}\frac{\partial^2 P}{\partial q_\mu \partial q^\mu} + \frac{1}{2}\frac{\langle \delta Q_\mu \delta Q^\mu \rangle}{\delta t}\frac{\partial^2 P}{\partial Q_\mu \partial Q^\mu}.\end{aligned}$$

It is important to note that the sum over indices in the last two terms is over all four indices, not the usual two. Once more we see the violation of Liouville's theorem via the coupling to the heat bath.

From this it is straightforward to obtain the equations governing the quadratic moments. Following the global method and defining τ, u, v, w as before, plus

$$x = \langle q^2 \rangle/\eta^2 \quad , \quad y = \langle q^\mu Q_\mu \rangle/\eta^3 \quad , \quad z = \langle Q^2 \rangle/\eta^4 \tag{23}$$

we arrive at

$$\begin{aligned}\dot{u} &= 2v + \delta_1, & \dot{x} &= 2y + \delta_3, \\ \dot{v} &= -f_2 u - \tfrac{\lambda}{2}u^2 + e^2 x u + w, & \dot{y} &= -\left(\tfrac{k^2}{\eta^2}\right) x - 2e^2 u x + z, \\ \dot{w} &= -2f_2 v - \lambda u v + 2e^2 x v + g u \left(\tfrac{k^2}{\eta^2}\right)\tfrac{T}{\eta} + \delta_2, & \dot{z} &= -2\left(\tfrac{k^2}{\eta^2}\right) y - 4e^2 u y + \delta_4,\end{aligned} \tag{24}$$

where

$$f_2 = \left(\frac{k^2}{\eta^2} - \frac{\lambda}{2} + \frac{\tilde{\lambda}}{2}\frac{T^2}{\eta^2}\right), \tag{25}$$

and δ_1, δ_2, δ_3 and δ_4 are the thermal fluctuation terms. Note that the terms involving Δ_μ have integrated to zero. Clearly we are not going to get too far with an analytic approach this time, so we restrict ourselves to a numerical analysis.

As before, our first preparation is to calculate the fluctuation terms. Since the coupling of ϕ to ψ is independent of gauge fields, the two new fluctuation terms, δ_3 and δ_4 must be zero. Hence, the only non-zero fluctuation term is δ_2, which is unchanged.

We consider once more an initial domain structure of length scale ξ_G, and take $e^2 = 40\lambda/3$, since we expect the gauge coupling to dominate. Figure 3 shows the effect of varying λ and g, the results being very similar to those observed in the global symmetry case. Figures 3a and 3b once again demonstrate the effect of increasing the size of the fluctuations; an increase in the rate of growth of the lower bound and a damping of oscillations, whilst Fig.3c reveals how decreasing the size of the self-coupling decreases the frequency of oscillations, the effect of fluctuations decreasing in a similar manner (Fig.3d).

The most important feature however is that, as in the global case, any non-trivial domain structure present at t_G is seen to be stable against fluctuations at greater times. This reinforces the work by Brandenberger and Davis[11], and, similarly, the arguments in favour of the Kibble mechanism.

4 Including the Expansion of the Universe

Until now we have ignored the expansion of the Universe, which we would expect to damp the amplitude of any oscillations present. To make the analysis more realistic it is necessary to include this expansion.

4.1 Global Symmetry

Taking first the case with a global symmetry, such a modification is straightforward, and leads to an extra term in the equation of motion for ρ proportional to the Hubble parameter H,

$$\partial_\mu \partial^\mu \rho - \partial_\mu \alpha \partial^\mu \alpha \rho + \frac{dV(\rho^2)}{d\rho^2}\rho = -3H\frac{\partial \rho}{\partial t} : \tag{26}$$

as expected, a damping term. The only effect this has on the equations for the quadratic moments u, v and w, is to add $-3Hv$ to the right hand side of equation (13), and $-6Hw$ to that of (14), though f_o is now written as

$$f_o \;=\; \left[\frac{k^2}{\eta^2}\left(\frac{a_0}{a}\right)^2 - \frac{\lambda}{2} + \frac{\tilde{\lambda}}{2}\frac{T^2}{\eta^2} - \frac{(k_\alpha \alpha)^2}{\eta^2}\left(\frac{a_0}{a}\right)^2\right] \tag{27}$$

where a is the expansion parameter, and a_0 its value at T_G[1].

Figure 4 shows the results for a small selection of values to illustrate the effects of varying the different parameters. Once more we consider an initial domain structure of length scale ξ_G. All four diagrams are seen to display the rapid damping due to the expansion of the Universe.

Figs. 4a and 4b demonstrate the effect of fluctuations. In 4b, where the fluctuations are suppressed, the lower bound on u rises much more slowly than in the unsuppressed case, Fig.4a, which actually overshoots its asymptotic value of one before reconverging. Hence, it is seen that fluctuations actually make it less likely that a configuration will be erased, agreeing with our non-expanding simulations. The upper bound varies very little between the two.

Fig. 4c shows the effects of reducing the self-coupling; a longer period of oscillation, a less dramatic initial growth and a much gentler approach toward its asymptotic value. Fig. 4d demonstrates how for small values of λ the fluctuations have very little effect on the evolution of the fields.

In summary, topologically non-trivial configurations are stable to thermal fluctuations. We also note that due to the scale factor now present in the equations of motion, the effect of k_c and k_α is rapidly damped out so that in all expanding cases considered, u converges on one, corresponding to $\langle \rho^2 \rangle$ tending to η^2, the long-time minimum of the effective potential.

4.2 Local Symmetry

Now including gauge fields once more, the equation of motion for A_μ is, predictably, modified in a very similar way to that for ρ when we include expansion, the new version being

$$\partial_\mu \partial^\mu A^\nu + 2e^2\rho^2 A^\nu - 2e\rho^2 \partial^\nu \alpha \;=\; -3H\partial_0 A^\nu. \tag{28}$$

[1] Since $a(t)$ is an unphysical quantity, we can without loss of generality take $a_0 = 1$.

In addition, the equations for the quadratic moments x, y and z acquire near identical terms to those already acquired by those for u, v and w, the only difference being an extra factor of two in the former case.

Once more, we illustrate four different values of the parameters, for a configuration varying on a correlation length scale. As in the non-expanding case, we take $e^2 = 40\lambda/3$ throughout.

Figures 5a and 5b demonstrate the effect of fluctuations. In Fig.5a, where the fluctuation coupling is comparable to the self-coupling, the lower bound to fluctuations is seen to rise much quicker (as was the case for a global symmetry) than that in Fig.5b where the fluctuations are suppressed. So much so in fact that it overshoots its expected limit, though long time studies show that it gradually bends back and converges to one.

In Fig.5c we see the effect of decreasing the self-coupling; a less dramatic growth of the lower bound, whilst Fig.5d reveals, once more, that the effect of fluctuations decreases dramatically with λ. Much the same as in the previous three cases.

Comparing figures 4 and 5, what we notice is that the presence of gauge fields heavily damps the initial growth leading to a lower upper bound and consequently smaller oscillations.

5 Conclusions

By studying the Langevin equations for the classical fields we have verified that for a U(1) model with broken global symmetry (whether expanding or not), string configurations formed during a second order phase transition, are stable to thermal fluctuations below the Ginzburg temperature. We have also shown this to be true in the case of a system with a local symmetry, reinforcing earlier work[11] on the subject, and lending further support to the Kibble mechanism for the formation of topological defects.

The same method has also been used to study how the fields evolve at late times, with the scalar field gradually tending to its equilibrium value; a process accelerated by the damping produced by an expanding universe. Indeed, our method tracks the evolution of the field *throughout* the phase transition. Other methods are only able to consider early or late times. In addition, we have seen that thermal fluctuations actually accelerate the early evolution of the field, and damp the amplitude of oscillations in the field as it tends to its asymptotic value, making it even less likely that a fluctuation will destroy a configuration.

This work is still an approximation however, since we have had to assume $\dot{\alpha} = 0$, leading to an arbitrariness in the asymptotic value of the field in the non-expanding models. However, this should not affect the stability of configurations, and in the expanding case this problem is smoothed out anyway as the model scales with time.

Another approximation we have made is in neglecting the dissipation term necessary when a source of fluctuations is present[13][14]. Since any dissipation term would have a damping effect, it would only increase the stability of a non-trivial domain structure, and so including it, a further avenue of research, should only strengthen our results. This and the quantum field theoretical treatment are in progress[15]

Finally, the flexibility of the Langevin equation approach[12] may make this method suitable for a number of other applications, such as the study of the evolution of seed magnetic fields following the breaking of a non-Abelian symmetry[16], and the stability of defects to

fluctuations in condensed matter systems such as ^{4}He[17]. Work on these subjects is in progress[18].

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